Perturbations in the Cosmic Microwave Background caused by excited Quantum States and Connection to Observation

Presented by Jason Malnar

In partial fulfillment of the requirements for graduation with the
Dean's Scholars Honors Degree in Physics and Astronomy

May 15, 2013

Abstract

I studied the effect of excited state initial conditions on the inflationary perturbation wavefunction and the possible effects they have on the observed spectrum of primoridal fluctuations. A de Sitter Universe approximation was used with a quantum harmonic oscillator model for the initial conditions in the Schrodinger picture. It was found that the power spectrum of any excited state only changes by a scalar multiple. This tells us the amplitude of the fluctuations are different for each corresponding state, which causes an observable change in the power spectrum. For the super-horizon modes of the fluctuations, $k/a(t) \ll H$, resulting from the process of inflation, a spectral index of $n_s = 1$ was found.

1 INTRODUCTION

The Universe we observe today is filled with thermal radiation in the form of an almost isotropic blackbody spectrum known as the Cosmic Microwave Background (CMB). The CMB is a direct record of the conditions of the Universe $\sim 380,000$ years after the Big Bang. The conditions of the Universe at this time have evolved to the large scale structure of the Universe that we observe today, and the conditions at the time the CMB was created are a result of the evolution of the Universe from the beginning up to that time.

The observation of the CMB reveals that the Universe has a background temperature of $\sim$ 2.725 K [4]. However, there are very small variations (on the order of a hundred thousandth of a percent or 10^{-5} K) throughout the temperature distribution across the sky. These variations are caused by quantum fluctuations in energy density of the very early Universe, and were magnified to macroscopic scale during a period of accelerated expansion of the Universe known as inflation. The variations observed in the CMB, known as anisotropies, reveal a great deal about the constituents of the Universe and are thought to have grown into the large scale structure of the Universe that we know today.

It has been previously believed that Inflation erases all traces of initial conditions, and thus, the spectrum of fluctuations in the CMB are insensitive to these initial conditions. However, recent work has shown that it might be possible for certain initial conditions to have an impact on the formula describing the spectrum of fluctuations in the CMB which could lead to observational confirmation. See for example [7].

In this paper I attempt to determine if the initial conditions, specifically excited quantum states, can have an observationally verifiable effect on the fluctuations in the CMB. I will determine what effects the initial conditions will have on the perturbations in the CMB and compare the results of this model with observational data. I will use the Schrodinger picture and a quantum harmonic oscillator model to describe the perturbations in the inflationary wavefunction. Using the initial excited state wavefunction, one can propagate it through time using a single field, slow-roll inflationary model and a de Sitter approximation, which I will discuss later in this paper. There are many comparisons one can make, but the most common is a calculation of the spectral index (or tilt) of the power spectrum. This parameter has been measured by WMAP and Planck and has been constrained by these observations.

It has been found that having an initial excited quantum state for the perturbations will only modify the amplitude of the power spectrum present in the CMB, and not the tilt. The implications of these results will be discussed later, however, this shows that the initial conditions can have an observable effect on the amplitude of the CMB power spectrum, and can thus be observed.

In section 2 I will include a basic overview of the Cosmology needed to understand the main idea presented in this paper. In section 3 I will discuss the observational results obtained with Planck 2013. In section 4 I will describe the methods used to calculate the power spectrum as well as a brief description of the initial conditions used in the calculation. I will also give the results of my calculations. Section 5 compares my results to the Planck data and finally, in section 6 I give my conclusions.

2 BACKGROUND

Cosmology is the study of the large scale structure of the Universe in its entirety. The goal is to understand the origin and evolution of the Universe, as well as to make predictions of the eventual fate of the Universe. The field of cosmology has become one of the most active scientific fields in modern times, however the question of what the Universe is and how it started has been contemplated for thousands of years.

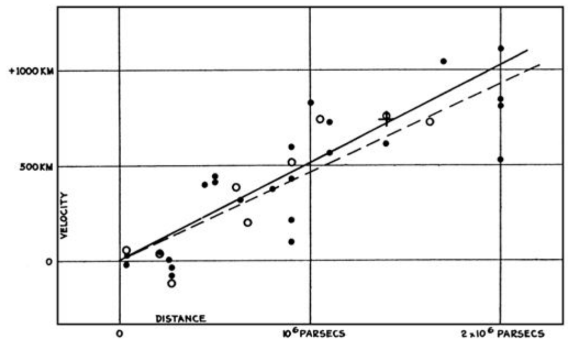

Figure 1: Velocity-distance relation found by Hubble in 1929. These are radial velocities corrected for solar motion. The circles represent the observed galaxies and the lines represent the linear relation and the corrected linear relation [9].

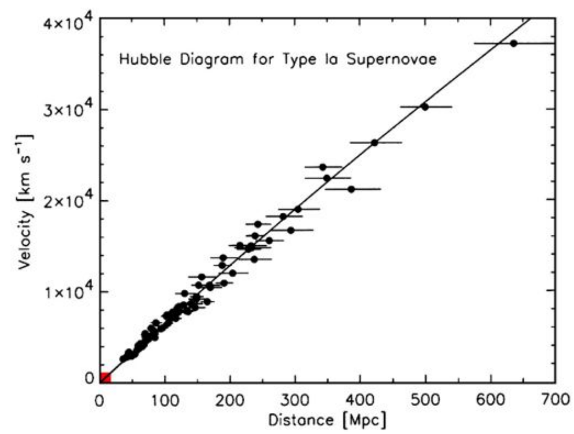

Figure 2: The Hubble diagram for type Ia supernovae. From the compilation of well observed type Ia supernovae by S. Jha. The scatter about the line corresponds to statistical distance errors of $< 10\%$ per object. The small red region in the lower left marks the span of Hubble's original Hubble diagram from 1929 [9].

There have been many theories on the origin and evolution of the Universe, however, the most widely accepted theory is the *big bang theory*. Great advancements have been made in the observational techniques used in cosmology, which have made it possible to test and confirm many of the predictions made by the big bang theory.

2.1 BRIEF HISTORY OF THE UNIVERSE

Edwin Hubble, an Astronomer at the Mount Wilson Observatory, made one of the most significant observational discoveries in history. He discovered that not only is our Universe much larger than previously thought, but it is also expanding in all directions. Along with Milton Humason, they photographed the spectra of many galaxies, and by observing the apparent brightness and the pulsation of Cepheid variable stars in these galaxies, they were able to measure the distance to each galaxy. However, this was not the most significant part of their research. Hubble and Humason also found that almost all of the galaxies showed a redshift in their spectrum, which signified a receding velocity. In other words, all of these galaxies were moving away from us. In fact, the further away the galaxy was, the larger the redshift. A direct correlation was determined between the distance to the galaxy and its observed redshift. This is called the **Hubble law** and is stated by the following formula [1]:

$$v = H_0 d \tag{1}$$

where v is the recessional velocity of the galaxy, d is the distance to the galaxy, and H_0 is known as the Hubble constant. Current observations have set the value of the Hubble constant at $H_0 = 67.15 km/s/Mpc$ [12]. This discovery was the most important contribution in establishing a theory of an expanding Universe.

Following the discovery of an expanding Universe, there have been many more important discoveries and theories that have led to the creation of what is now called the **Standard Model of Cosmology**. This standard model is characterized by and built on four main parameters that have been observationally measured quite accurately. The current expansion rate of the Universe H_0, the mean matter density of the Universe ρ_m, the background temperature of the Universe

T_{CMB}[1], and the density of the vacuum energy which is described by the cosmological constant Λ. The cosmological constant was initially introduced by Einstein and his theory of General Relativity with the purpose of allowing for a stationary model of the Universe. However Λ is now interpreted as the energy density of the vacuum.

The current standard model is also called the ΛCDM model. Basically, this is a model containing the cosmological constant Λ and cold dark matter (CDM).

The standard model of cosmology is also based on two postulates. First, our place in the Universe is not special, and not distinguishable from other locations in the Universe. Second, the distribution of matter in the Universe is isotropic on large scales. This leads to a homogeneous and isotropic model called the **Friedmann-Lemaitre model**. According to this model, the Universe used to be smaller and hotter in the past and it has continuously expanded and cooled down to the state it is presently in.

2.1.1 THE BIG BANG

If the Universe is expanding and has a finite age, it is conceivable to think that if you reverse time the Universe will shrink and become more and more dense, until it reaches what is called a *singularity.* This initial state of the Universe, or beginning is called **the Big Bang Singularity**. The Big Bang is thought to be the originating event of the Universe, and also caused the Universe to expand over the 13.81 million years [12] since the Big Bang occured.

Big Bang Cosmology is based on the homogeneity and isotropy of the Universe as well as Einstein's theory of General Relativity. The success of this theory has been shown through a large number of its predictions being observationally verified.

2.1.2 DYNAMICS OF EXPANSION

The idea of the expansion of the Universe is somewhat misleading. The galaxies are not physically receding from each other in space (even though they do move through space), the space itself is actually expanding. Like a large rubber sheet being stretched out. Therefore, the laws of physics and the mathematics describing normal motion had to be adjusted to account for the expansion of space.

If we want to measure distance through space in a Universe for which space itself is expanding we have to include another factor to account for the expansion of space. This is called the *scale factor* and is defined as $a(t)$. For example, if we want to measure the distance between two galaxies in an expanding Universe and see how the distance changes through time due to that expansion we would use this equation: $d(t) = a(t)d_0$

The more appropriate way to describe the geometry of space is to use a metric, which resembles the distance equation $s^2 = x^2+y^2+z^2$. Therefore the change in position, or distance is described as $ds^2 = dx^2+dy^2+dz^2$. Now one needs to know how to describe the geometry of our Universe, which is homogeneous and isotropic, and includes the time coordinate introduced by Einstein in special relativity. This is done with the *Freidmann-Robertson-Walker* (FRW) metric generally written as [8]

$$ds^2 = -dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega\right] \tag{2}$$

where K is known as the curvature parameter of space and is a constant defined as

[1] T_{CMB} refers to the extremely uniform temperature of the CMB. This temperature helps determine the radaition density ρ_R.

$$K = \begin{cases} +1 & spherical(closed) \\ -1 & hyperbolic(open) \\ 0 & Euclidean(flat) \end{cases}$$

This formulation is also known as *FRW spacetime*.

Figure 3: Visualization of the geometry of the Universe. $K = 1$ refers to the spherical Universe on the left. $K = -1$ refers to the hyperspherical Universe in the center. $K = 0$ refers to the flat Universe on the right [15].

Albert Einstein created the theory of General Relativity to describe the geometry of the Universe and the gravitational interaction between massive objects due to the warping of spacetime. The Einstein field equations give us the mathematical framework through which the expansion of the Universe can be explained. One simple version of Einstein's equation is given by

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{3}$$

where $G_{\mu\nu}$ is the *Einstein tensor* which describes the curvature of spacetime, $T_{\mu\nu}$ is the *stress-energy tensor* describing the density and flux of energy and momentum in spacetime, and G is *Newton's gravitational constant* [2].

The Friedmann Equations are a closed set of solutions to Einstein's equations. They are used in the standard model of cosmology to describe the expansion of space in the context of general relativity by describing how the scale factor $a(t)$ evolves.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho c^2 + 3P) \tag{5}$$

Here ρ is the energy density of the Universe, P is the pressure and K is defined above [8]. In fact, these equations are Einstein's equations for an FRW spacetime filled with a perfect fluid. This is where the thermodynamics comes into play, and where the pressure term comes from.

The results of Hubble's observations can be related to the Friedmann equations by the **Hubble parameter**:

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{6}$$

The Hubble constant H_0 can be calculated from these equations by inputing time $t = t_0$ which corresponds to the present time.

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \tag{7}$$

2.1.3 THERMAL HISTORY

Now that we know our Universe started at an extreme temperature, above 10^{31} Kelvin (speculatively), and is presently 2.7 K on average, we can see there exists an extreme thermal range throughout the evolution of the Universe. Assuming the laws of physics have been the same throughout the entire evolution, one can assume there have been many different states of the Universe corresponding to various temperatures or energies. Using General Relativity and the laws of physics and thermodynamics, one can predict the state of matter in the Universe at any time in history, and knowing the constituents of the Universe and the expansion rate, one can calculate how the Universe has changed, or evolved through various epochs in its history.

In its earliest stages the Universe was filled with something called the *primordial soup*. This was an extremely hot and dense plasma full of fundamental particles. The energy of these particles at this time was high enough to prevent atoms and nuclei from forming, thus leaving a mixture of quarks, leptons and gauge bosons (or force carrier particles like the photon). As the Universe expanded and cooled, the energy droped and allowed for nuclei and later atoms to form, and so on. Below is a chronological progression of the Universe from beginning to present time.

- **The Planck Epoch**: A period immediately after the Big Bang up until about 10^{-43}s. It is during this period that the distances in the Universe were less than the Planck length, which is the length at which the structure of spacetime becomes dominated by the quantum effects of gravity. However, there is no accepted theory of quantum gravity[2] so science is not able to make predictions about the events occurring during this time. Therefore, this represents a time for which our knowledge of the conditions of the Universe is extremely limited.

- **Inflation**: This is speculated to be a time of expansion dominated by a slowly varying vacuum energy Λ that caused the scale factor to grow exponentially [2]. During this extremely short period of Inflation, it is believed that the Universe grew by a factor of about 10^{50} in only a period of about 10^{-32}s [1].

- **Reheating**: Through the extremely accelerated expansion of inflation both the density and temperature of the Universe droped greatly. As inflation came to a stop, all of the energy involved in this acceleration was transfered back to the matter/radiation in the Universe, thus heating the Universe back up.

- **Neutrino Decoupling**: Approximately 1 second after the Big Bang, the Universe had cooled to around 10^{10} K and the neutrinos were no longer able to interact with the rest of the plasma. They were now freely propagating through the Universe.

- **Big Bang Nucleosynthesis (BBN)**: As the Universe continued to expand and cool, there came a time when the energy of the particles was low enough to allow for protons and neutrons to bind together and form complex nuclei. First, deuterium was produced, which further combined to form tritium and helium. At this time, almost all of the neutrons were bound into helium-4 (^{4}He), with a very small amount of deuterium, ^{3}He, and ^{7}Li present. The predicted abundances of these elements have been observationally measured and agree well with the predictions made in the standard model of cosmology. However, there existed

[2] The theory of general relativity fails on the quantum scale. There is no accepted theory successfully combining general relativity and quantum mechanics.

more protons than neutrons at this time (due to the mean lifetime of a neutron being about 15 minutes), so after all the neutrons were used to form nuclei, there was a leftover abundance of protons.

- **Recombination**: Nearly 380,000 years after the Big Bang the temperature had dropped to approximately 3000 K. This allowed electrons to bind to the free protons to form hydrogen by $p + e^- \to H + \gamma$. This effectively ended the period when the Universe was composed of an electrically charged plasma and was now mostly a neutral gas. Up until this moment, the Universe had been mainly composed of free electrons and protons, helium nuclei, photons (and a small amount of non-interacting dark matter). The photons had been interacting with the charged particles, continually scattering off the electrons. The energy of the Universe was now low enough to allow for the electrons and protons to form neutral atoms, thus allowing the photons to decouple from electron scattering and begin to travel essentially without interaction. This moment in time is known as *decoupling*, since the photons were no longer scattered by the charged particles. These unimpeded photons form what is now called the **Cosmic Microwave Background**, and what we see in the CMB is referred to the *surface of last scattering*. This surface was created because all of the photons stopped scattering almost simultaneously and these propagating photons created something like the surface of a sphere. The prediction of the Cosmic Microwave Background and its discovery is the biggest piece of evidence in support of the standard model of cosmology.

- **Dark Ages**: This is a period of time thought to have lasted from the end of recombination to about 200 million years [8] after the Big Bang. During this time the Universe was dark, meaning no light was being produced since no stars yet existed. The matter in the Universe was slowly clumping together under the attraction due to gravity and the eventual formation of stars signifies the end of this epoch.

- **Reionization**: As stars formed from the gravitational collapse of dense regions of matter, they began producing intense radiation (some in the form of visible photons). This radiation essentially reionized the surrounding Universe, and the Universe was again mostly composed of plasma.

- **Galaxy Formation**: Approximately 500 million years after the Big Bang, the first galaxies started to appear [8]. These galaxies also began to clump together to form the largest objects in the Universe; clusters and superclusters. These objects make up the large scale structure of the Universe we see today.

Table 1: History of the Universe [1][5][8]

Period	Time	Energy	Temperature (K)
Planck Epoch	$< 10^{-43}$s	10^{18} GeV	10^{31}
Inflation	$\geq 10^{-34}$ s	$\leq 10^{15}$ GeV	
Neutrino Decoupling	1 s	1 MeV	10^{10}
BBN	3 min	0.1 MeV	
Recombination	380,000 years	1 eV	10^4
Dark Ages	10^5 - 10^8 years		
Reionization	$\sim 2 \times 10^8$ years		
Galaxy Formation	$\sim 5 \times 10^8$ years		

2.2 THE COSMIC MICROWAVE BACKGROUND

The Cosmic Microwave Background (CMB) is the oldest observable radiation in the Universe. It is a background of microwave radiation that originated as a result of the process of recombination and fills all of space. As a result of the free electrons, protons, and helium nuclei combining to form neutral atoms, the photons in the Universe were allowed to freely propagate from whatever configuration they were in. Since then, these photons have traveled almost completely unchanged to the present time. Therefore, the CMB is a direct record of the state of the Universe at the time of recombination 380,000 years after the Big Bang.

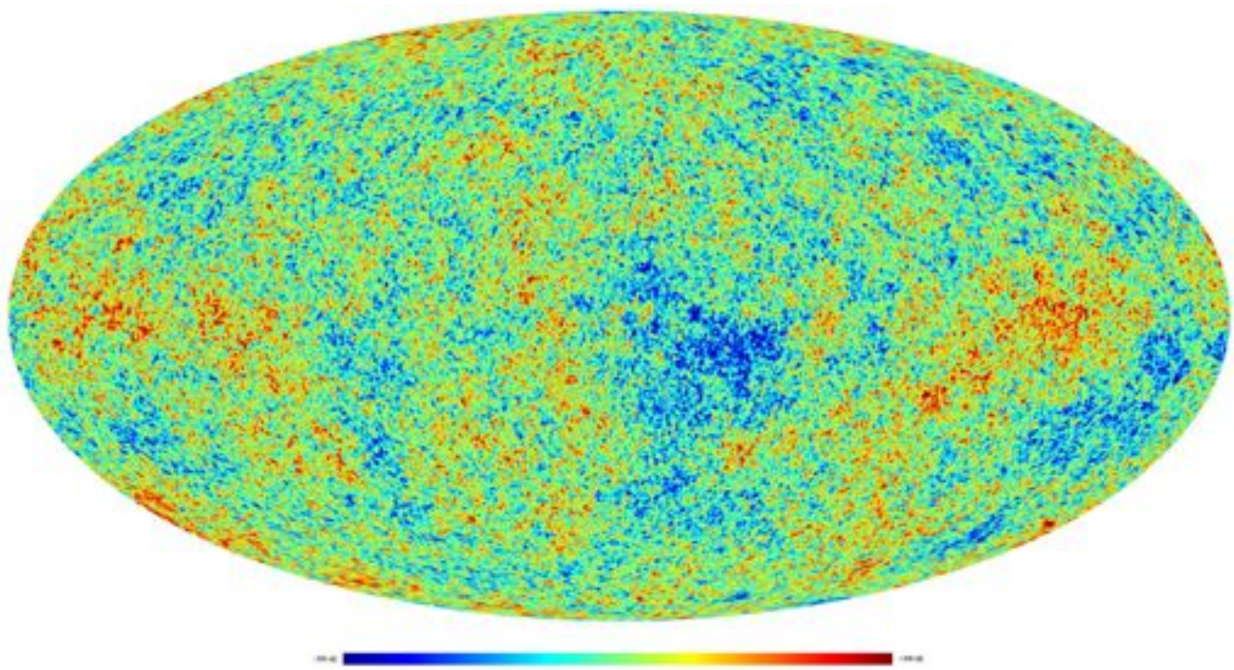

Figure 4: CMB sky map produced by the Planck 2013 results [11]. You can see the scale of the temperature variations on the bottom.

Since the Universe is isotropic on large scales, the CMB should conceivably be isotropic as well, and it is to a remarkable degree. The CMB pictured above is a measure of the temperature of the background radiation which is approximately $2.75K$. As you can see in figure (4) there are variations in the background temperature, however, they are extremely small, varying only by about 10^{-5} K throughout the entire sky [8].

These temperature fluctuations in the CMB represent small density fluctuations present in the Universe at the time of recombination, and these density fluctuations are believed be leftover from quantum fluctuations during the period of inflation. Through inflation, these quantum fluctuations grew with the Universe, causing the temperature fluctuations in the CMB, and they have further grown with expansion to eventually form the large scale structure of the Universe present today. Thus, a study of the CMB is in effect a study of the mechanism that formed the Universe as we know it today.

2.3 PROBLEMS WITH THE BIG BANG THEORY

The Big Bang Model has achieved great success and is widely accepted throughout the scientific community. However, there are still some aspects of the theory which are flawed and have received further attention. We will find out shortly that the theory of inflation successfully solves these problems.

2.3.1 THE HORIZON PROBLEM

It has been shown previously that the CMB is nearly perfectly isotropic, however, given the present size and age of the Universe, this seems almost impossible. Here on Earth, we can only see objects that lie within our *cosmic light horizon* which is a sphere centered on the Earth with a radius equal to the maximum distance light could have traveled in the 13.8 billion year lifetime of the Universe. Any object outside of this horizon can not be seen because the light traveling from that object has not had enough time to reach Earth (See figure (5)).

Now imagine two regions in the Universe existing on the edge of our cosmic light horizon, but in opposite directions from Earth. The spatial seperation of these two regions is much greater than each regions respective particle horizon, and thus, the two regions cannot "see" each other, or in other words, they do not share information with each other. This is known as *causal connection.* If there are many regions throughout the Universe that are not causally connected, how is it that the CMB is so remarkably uniform throughout the entire Universe? If these regions in space cannot share information, they shouldn't be expected to have almost exactly the same temperature. This is known as the horizon problem.

2.3.2 THE FLATNESS PROBLEM

The observed flatness of the Universe presented scientists with another problem. The geometry of our Universe depends on the density parameter

$$\Omega_0 = \frac{\rho_0}{\rho_c} \tag{8}$$

as seen in equation (2), where ρ_0 is the combined average mass density and ρ_c is the critical density that determines the geometry of the Universe. If $\rho_0 = \rho_c$, $\Omega_0 = 1$ and $K = 0$ and the Universe is considered to be flat, as seen in figure (3). Recent data from Planck measured the curvature to be very nearly zero [12].

In order for $\Omega_0 \approx 1$ today, it must have also been extremely close to 1 during the Big Bang. In fact, any slight deviation from this value would have caused the Universe to either expand so rapidly that no structures would have formed, or be so dense as to collapse back on itself. Since we do live in a flat Universe, we know that immediately after the Big Bang ρ_0 must have been equal to ρ_c to within 50 decimal places [1]. This means that an extremely precise amount of fine-tuning was needed to create the Universe we know today. How does our Universe contain an average mass density that is equal to the critical density to such a remarkable degree? This is known as the flatness problem.

2.4 INFLATION

The idea of an exponential expansion, or inflation, in the very early Universe was hypothesized by such authors as Starobinsky, Kazanas and Sato. However, the idea did not take hold until the early 1980's when Alan Guth further described inflation as a mechanism to solve some of the problems in cosmology. Later, Linde, Albrecht and Steinhardt improved on Guth's model forming the current theory of inflation.

In this inflationary model, it was presumed that in the very early Universe the vacuum energy density (Ω_Λ) was much higher than today, and this made it the dominant driving factor of the expansion of the Universe. Meaning, during this period, the expansion of the Universe can be approximated as an expansion due only to the vacuum energy and all other sources can be neglected. There is a model based on this assumption called the *de-Sitter Universe*, which will be discussed in more detail later.

The exponential expansion of the Universe during inflation makes it possible for the entire visible Universe to have been causally connected prior to, and during the inflationary period. In other words, every region of the Universe was able to come to thermal equilibrium because of their close proximity, and after Inflation, the Universe spread these regions out to distances outside of their particle horizons (see figure (6)).

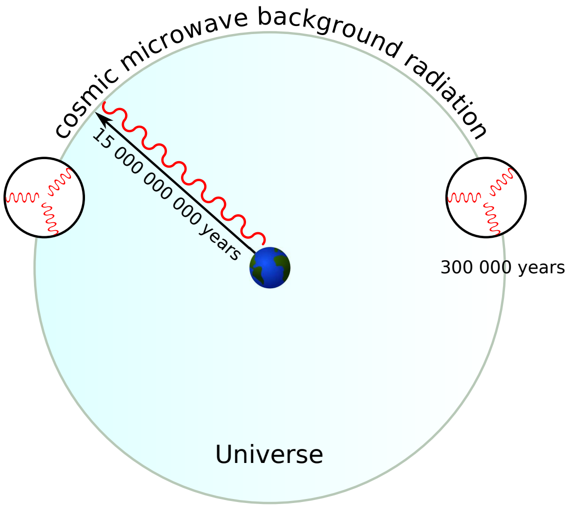

Figure 5: When we see the CMB we are looking at radiation from when the Universe was only about 300,000 years old. At that time, the radiation only had enough time to travel to distances within the smaller circles. However, the physical size of the CMB is much larger and we can see the two small circles are not in causal contact.

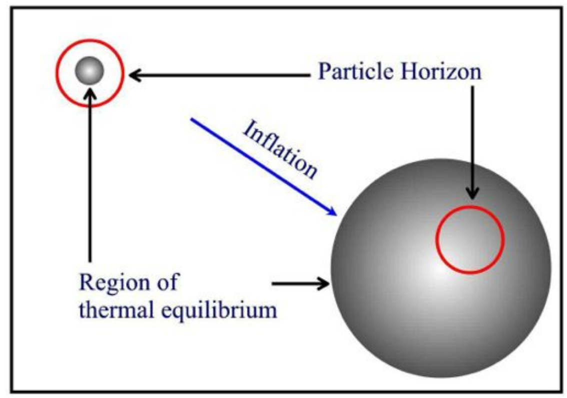

Figure 6: During inflation regions that have achieved thermal equilibrium can be expanded outside the particle horizon (Hubble distance). After inflation, the particle horizon begins to expand faster than the spacetime and these regions reenter the horizon. Inflation is the only known way to explain this uniformity, thus solving the horizon problem [16].

Another success of the theory of inflation is in solving the flatness problem. In order to get rid of the fine-tuning needed for an extremely flat Universe one must think of a possible way for the Universe to be initially curved but presently flat. Through the accelerated expansion of space during Inflation, the Universe grew from a total size of less than that of an atom, to the size of our solar system. For example, as a sphere is inflated, its curvature eventually becomes undetectable and the surface of the sphere will appear flat. Likewise, Inflation stretched the Universe out enough for the curvature to be undetectable. See figure (7).

In the pre-Inflationary period, the Universe was extremely small; on the scale of quantum mechanics. The Heisenberg uncertainty principle tell us that the matter distribution cannot be perfectly homogeneous, and thus quantum fluctuations existed. The accelerated expansion of the Universe during Inflation took these microscopic fluctuations and blew them up to macroscopic scale fluctuations. Thus, the large scale structure of the Universe is an extremely magnified representation of the quantum fluctuations in the very early Universe. The importance of these fluctuations is appreciated by the fact that if they did not exist, the Universe would be exactly the same everywhere and no physical matter would have been able to clump together through gravitational attraction.

The simplest model of inflation that fits the observational data involves a single scalar field ϕ, which is known as the *inflaton* [11]. This inflaton field will have a corresponding potential energy $V(\phi)$ and kinetic energy $\frac{1}{2}\dot{\phi}^2$. Assuming a spatial homogeneity ($\phi(x,t) = \phi(t)$), we can get a general

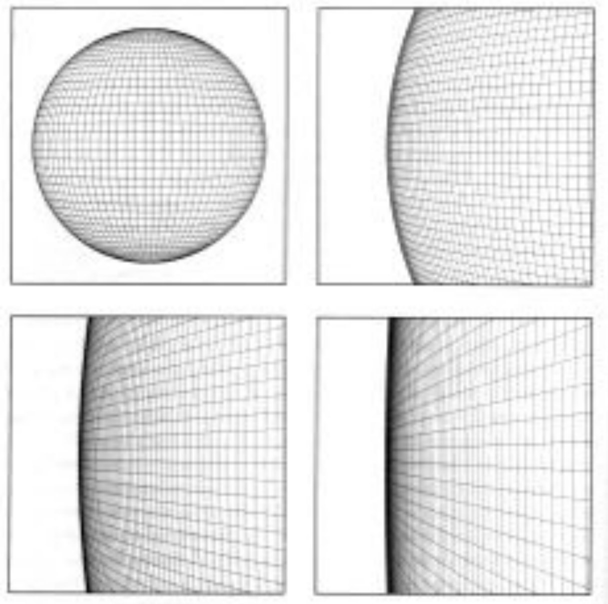

Figure 7: As the Universe expands, the curvature on smaller scales become unnoticeable [10].

Lagrangian for the inflation field:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{9}$$

Thus, one can obtain the equations of motion for the inflationary field for any shape of the inflaton potential. One can also visualize the inflaton with a potential energy graph (see figure 8).

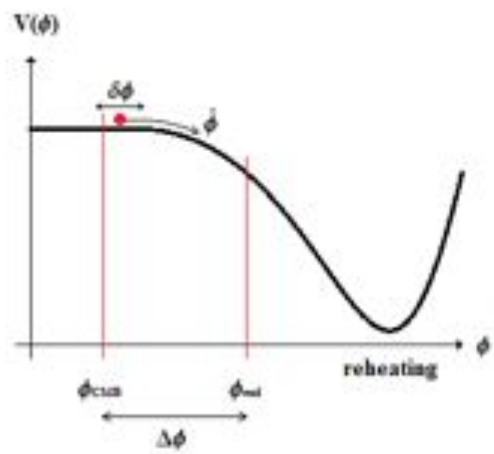

Figure 8: Example of an inflaton potential. Acceleration occurs when the potential energy of the inflaton field, $V(\phi)$, dominates over the kinetic energy of the field, $\frac{1}{2}\dot{\phi}^2$. Inflation ends when the kinetic energy is comparable to the potential energy, $\frac{1}{2}\dot{\phi}^2 \approx V(\phi)$. At reheating, the energy density of the inflaton is converted into radiation [5]. This is also the potential representing the slow-roll inflaton.

If one thinks of a homogeneous background field $\bar{\phi}$, representing a perfectly homogeneous Universe, then the fluctuations can be thought of as small perturbations in the background field, $\delta\phi$. A convenient gauge typically chosen in this situation is $\delta\phi = \varphi(x,t)$. This helps to distinguish the background field ϕ from the perturbations [14].

$$\phi(x,t) = \bar{\phi}(t) + \varphi(x,t) \tag{10}$$

In order to account for curvature in this perturbation term the scalar curvature perturbation $\mathcal{R}$ is introduced. One can think of the evolution of the Universe as being described by a uniform density hypersurface Ψ at every instant in time. $\mathcal{R}$ measures the spatial curvature of these comoving hypersurfaces. The scalar perturbation can be related to the background field deviation by [5]

$$\mathcal{R} = \Psi + \frac{H}{\dot{\phi}}\varphi \tag{11}$$

It is this curvature perturbation field $\mathcal{R}$ that we are interested in studying and modeling, and will be working with throughout the rest of this paper. There are two main reasons why we want to work with $\mathcal{R}$. First, it is a gauge invariant quantity, and therefore an observable. Second, $\mathcal{R}$ is conserved outside of the horizon, meaning the conditions inside the Universe cannot affect $\mathcal{R}$ while it is outside the horizon. This is appealing because little is known about the physics of reheating, and since $\mathcal{R}$ is outside the horizon at this time, these perturbations are not affected by the physics of this epoch, or any period for which $\mathcal{R}$ remains outside the horizon.

2.4.1 SLOW-ROLL INFLATION AND THE DE SITTER APPROXIMATION

As mentioned earlier, the potential for a single field Inflation is well modeled by a very slow downward slope. This is called *slow-roll inflation.* If the potential is nearly flat, the field acts like a slowly varying vacuum energy. This was an unstable maximum, so as the Universe begain rolling down this potential, inflation started to happen. As the minimum in potential was reached, all the energy had been converted to matter/radiation in the Universe, and reheating began (see figure 8). Slow-roll inflation has the condition that $\dot{H}$ is close to, but not exactly zero. This model fits well with observation.

A de Sitter Universe is a further assumption made to the slow roll model. It models the Universe as spatially flat and neglects ordinary matter. Thus, the dynamics of the Universe are completely dominated by the cosmological constant or the inflaton field in the very early Universe. In slow roll inflation, $\dot{H}(t) \neq 0$. De Sitter expansion approximates this as completely flat, or $H(t) = H$, making it a constant and giving

$$\dot{H}(t) = 0 \tag{12}$$

Basically $\Omega_\Lambda \gg \Omega_m + \Omega_R$ lets us make an assumption that $\Omega_\Lambda = 1$. This will set the scale factor at

$$a(t) = e^{Ht} \tag{13}$$

2.5 HORIZONS

The maximum comoving distance light can travel in a certain period of time is called the **comoving particle horizon** [5], and was discussed briefly in the section on the horizon problem. Here I will discuss how this idea can allow for the quantum fluctuations in the early Universe to be conserved. The Hubble parameter has a unit of inverse time and therefore, the Universe can have a characteristic length-scale, called the Hubble length, of $d \sim cH^{-1}$. Thus, one can also define something called the *comoving Hubble radius* as $(aH)^{-1}$.

The physical horizon of the Universe plays an important role in the overall evolution of the Universe. We know that during inflation, the scale factor increases exponentially (see equation (13)) while the Hubble parameter remains nearly constant (due to slow-roll). This tells us that for a mode which begins inside the horizon it will be very far outside the horizon by the end of Inflation [8].

We can think of this in terms of either k or the wavelength. In the period before inflation, the physical momentum ($k/a(t)$) of the fluctuations, inversely proportional to the wavelength, is much larger than the Hubble length, or size of the Universe (or the wavelength is much smaller). During inflation, these fluctuations are stretched outside of the horizon. In other words, the physical momentum is now much smaller than the Hubble length (or the wavelength is much bigger).

- Before inflation the fluctuations are *inside* the horizon, otherwise known as the *sub-horizon* scale. See figure (9):

$$\frac{k}{a(t)} \gg H(t) \tag{14}$$

- During inflation the fluctations are stretched *outside* the horizon, otherwise known as the *super-horizon* scale. See figure (10):

$$\frac{k}{a(t)} \ll H(t) \tag{15}$$

Now, because inflation has caused the curvature perturbation field, $\mathcal{R}$, to move outside the horizon, no casual physics can affect it. Therefore the fluctuation remains conserved until it re-enters the horizon at a later time after inflation and during the normal expansion of the Universe. Another way to think of this is to say that the fluctuations are oscillating inside the horizon, but when they move outside the horizon the fluctuations are frozen and the oscillation stops.

Now we can give an important alternative definition of Inflation in terms of the horizon. Inflation is defined as a period in the very early Universe when the comoving Hubble radius, $(aH)^{-1}$, was decreasing [5].

3 PLANCK 2013 RESULTS

Planck is a collaborative mission between the Eurpean Space Agency and NASA to observe and analyse the CMB with the highest accuracy ever achieved[3]. This space telescope follows two previous telescopes with the same mission; COBE (Cosmic Background Explorer), launched in 1989, and WMAP (Wilkinson Microwave Anisotropy Probe), launched in 2001. The Planck telescope is able to measure the CMB with a temperature resolution on the order of one part in 10^6. Using the Planck results, Cosmologists were able to obtain more accurate values for the various parameter associated with the CMB. I will be comparing the results of my calculation to these results obtained by Planck. Figure (4) shows the sky map produced by the Planck results[4].

The main parameter I am going to compare to is the scalar spectral index (n_s), or tilt. Planck, combined with WMAP, determined a value for the spectral index of[5]

$$n_s = 0.9603 \pm 0.0073 \ (68\%; Planck + WP) \tag{16}$$

[3]http://sci.esa.int/science-e/www/area/index.cfm?fareaid=17

[4]All of the data in this section was taken from [12].

[5]Planck uses a scalar spectrum power-law index of $k_0 = 0.05$ Mpc^{-1}.

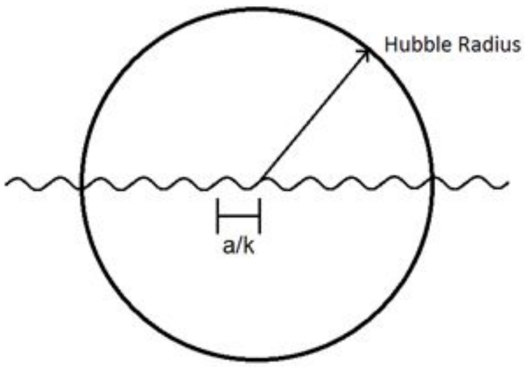

Figure 9: *Sub-horizon fluctuations.* Here we can see that the physical size of the wavelength is smaller than the physical size of the Universe, so the fluctuation is inside the horizon.

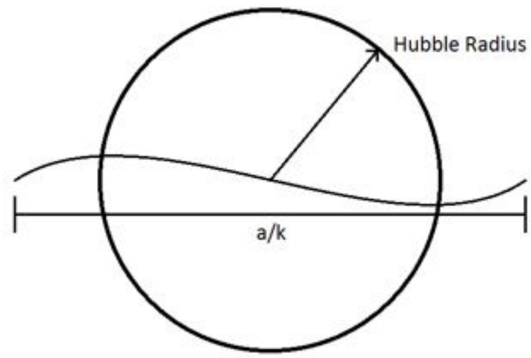

Figure 10: *Super-horizon fluctuations.* Here we can see that the physical size of the wavelength is larger than the physical size of the Universe, so the fluctuation outside inside the horizon.

This value agrees with $n_s < 1$ as expected in a single-field slow-roll inflationary model, and is also within the constraints $[0.9, 1.1]$ previously calculated for the standard model. Now I can compare my theoretical results with the range defining the accepted constraints, as well as with the Planck value for n_s.

The results from Planck also determine a relatively large deviation in the Hubble parameter noticeably lower than previous results. $H_0 = 67.3 \pm 1.2$ km/s/Mpc. In addition, the Planck results give values of $\Omega_b h^2 = 0.02205 \pm 0.00028$ for the physical density of baryons, $\Omega_c h^2 = 0.1199 \pm 0.0027$ for the physical density of cold dark matter, and $100\Omega_K = -0.10^{+0.62}_{-0.65}$ (95%; $Planck + lensing + WP + highL + BAO$). These new value will be used in my comparison.

Any variation in the spectral index can produce significant differences in a sky map as well as the angular distribution plot. For example, by changing the spectral index to $n_s = 1.1$ we see a definate increase in the amplitude of the spectrum, and in comparison, when we input a value of $n_s = 0.9$ we see a definate decrease (see figure 11).

4 CALCULATION OF THE POWER SPECTRUM

In performing this calculation a quantum mechanical wavefunction was used to represent the system of perturbations filling the entire Universe. Any system can be described using quantum mechanics, no matter the scale, but the larger the scale of the system the more complicated the quantum mechanics become. However, when the Universe was in the pre-inflationary period it was extremely small, making it necessary to use quantum mechanics.

The wavefunction, or Schrodinger picture, is not the only representation of quantum mechanics. There are other ways in which to perform calculations in quantum mechanics (i.e. the Heisenberg picture). We chose to use the Schrodinger picture because it is easier to conceptualize and visualize, and it gives a straightforward meaning of initial conditions as $\psi(t = 0)$.

For this calculation a Freidmann-Robertson-Walker (FRW) time and mostly a plus signature

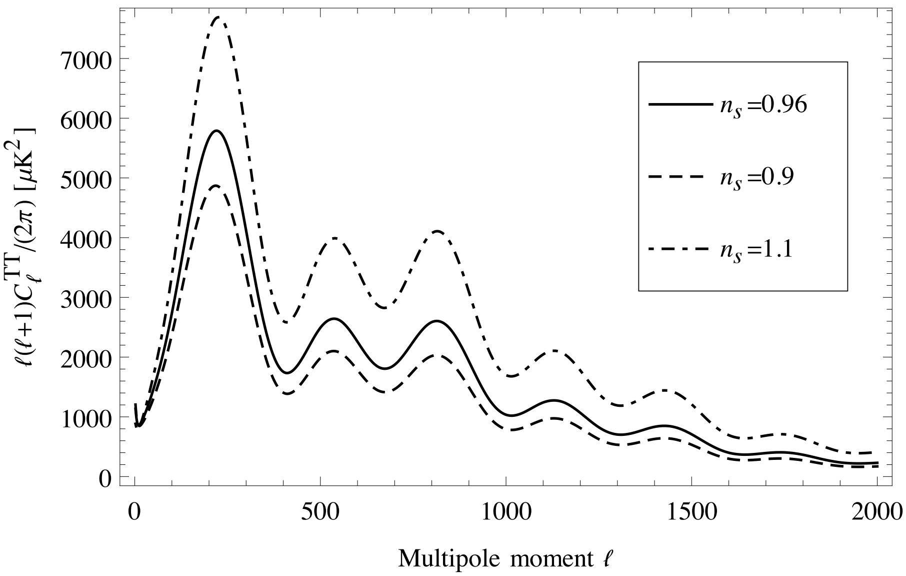

Figure 11: *Angular Power Spectrum* - This shows the noticeable difference in the angular power spectrum created from different values of the spectral index. These plots were created using the CAMB web interface on the NASA website [13].

was used. This gives the usual Robertson-Walker metric for this spacetime when $K = 0$ [7]:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \tag{17}$$

This is a study of a single field inflation model and the scalar fluctuations will be parameterized in terms of the comoving curvature perturbation, $\mathcal{R}_k$. Here, $\mathcal{R}_k$ is a gauge invariant quantity in k-space that is conserved outside of the horizon [7]. For the purpose of this paper, we only need to calculate the power spectrum of $\mathcal{R}_k$ by finding the expectation value of $\mathcal{R}_k^2$. Since we are using quantum mechanics $\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle$ is the quantum mechanical expectation value, and this depends on the choice of the state. Or in other words $\langle \psi_n | \mathcal{R}_k^2 | \psi_n \rangle = \langle \mathcal{R}_k \mathcal{R}_{k'} \rangle$

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = (2\pi)^3 \delta(k + k') P_{R_k}(k) \tag{18}$$

where $\mathcal{R}_k = \int d^3x \mathcal{R}(x) e^{-ik\cdot x}$ is the Fourier Transform from position space to k-space.

In a single field inflationary model one can consider the action of a scalar field minimally coupled to gravity. This can be expanded around a spatially homogeneous solution and the leading order term can accurately produce a Hamiltonian like that of a quantum harmonic oscillator [14]. The Hamiltonian for this case, when written in terms of $\mathcal{R}_k$, for every value of k is like that for a simple harmonic oscillator with a time-dependent mass and frequency [7]

$$\mathcal{H} = \frac{1}{2m(t)} \Pi_{\mathcal{R}_k}^2 + \frac{1}{2} m(t) \omega(t)^2 \mathcal{R}_k^2 \tag{19}$$

where $\Pi_{\mathcal{R}_k}$ represents the momentum conjugate to $\mathcal{R}_k$ and

$$m(t) = \frac{a^3(t)\dot{\phi}^2(t)}{H^2(t)} \tag{20}$$

$$\omega(t) = \frac{k}{a(t)} \tag{21}$$

There are some obvious advantages to having a Hamiltonian of this form. First, it can be solved exactly, and second, the free fields in curved spacetime, as well as de Sitter spacetime, are similar to a collection of harmonic oscillators, making this a viable option.

However, in this model we are using the de-Sitter approximation giving $\dot{\phi}(t) \equiv$ constant and $H(t) \equiv$ constant, and $a(t) = e^{Ht}$.

4.1 FINDING THE INFLATIONARY WAVEFUNCTION

A mathematical technique was developed in [6] that could be used to find the exact solution of the Schrodinger equation with a time-dependent mass and frequency. This is done by operating on the initial ($t = 0$) wavefunction with a unitary operator $U(t, 0)$ defined in [6].

$$\Psi(x,t) = U(t,0)\psi(x,0) = \int_{-\infty}^{\infty} dyK(x,t;y,0)\psi(y,0) \tag{22}$$

where $\psi(x, 0)$ is the initial wavefunction to be time-evolved, and

$$K(x,t;y,0) = \sqrt{\frac{i}{2\pi c_3}} \exp\left\{\frac{(xe^{c_2/2} - y)^2}{2ic_3} + \frac{ic_1}{2}x^2 + \frac{c_2}{4}\right\} \tag{23}$$

For this expression, all of the time dependency lies within the c_1, c_2, and c_3 terms as follows:

$$c_1(t) = m(t)\frac{\partial}{\partial t}\ln[F(t)] \tag{24}$$

$$c_2(t) = -2\ln|\frac{F(t)}{F(0)}| \tag{25}$$

$$c_3(t) = -F(0)^2\int_0^t \frac{du}{m(u)F(u)^2} \tag{26}$$

where an initial condition in this technique is given as $c_1(0) = 0$ and where $F(t)$ is obtained from the differential equation

$$\frac{d^2F(t)}{dt^2} + \xi(t)\frac{dF(t)}{dt} + \omega(t)^2F(t) = 0 \tag{27}$$

with

$$\xi(t) = \frac{\partial}{\partial t}\ln[m(t)] \tag{28}$$

and $m(t)$ given by equation (20).

We should expect a time-dependent wavefunction of the form below, which is an eigenstate of the Hamiltonian.

$$\Psi_k(\mathcal{R}_k, t) = G_k(\mathcal{R}_k, t)\exp\left[-\frac{1}{2}\mathcal{R}_kF_k(t)\mathcal{R}_{k'}\right] \tag{29}$$

Where $G_k(\mathcal{R}_k, t)$ represents the amplitude of the perturbation and $F_k(t)$ represents the width of the gaussian. You will see later that this width is important in the overall calculation of the power spectrum. In equation (29) we have transformed back to k-space, and again, this equation represents the state for each k, or each oscillator.

Once the initial conditions are chosen, the evolution of the power spectrum $P_{\mathcal{R}}(k)$, corresponding to the Hamiltonian in equation (19) can be found.

4.2 TIME EVOLUTION OF THE c_i TERMS

Here are listed the explicit expressions obtained for the $c_i(t)$ functions introduced in section 4.1.

$$c_1(t) = -\frac{a(t)^2 k^2 \dot{\phi}^2}{H^2(a(t)H + k\cot(\frac{k}{H} - \frac{k}{a(t)H})} \tag{30}$$

$$c_2(t) = -2\log\left[\frac{1}{a(t)}\cos\left(\frac{k}{H} - \frac{k}{a(t)H}\right) + \frac{H}{k}\sin\left(\frac{k}{H} - \frac{k}{a(t)H}\right)\right] \tag{31}$$

$$c_3(t) = \frac{H^2[a(t)H^2 + k^2 - Hk(-1 + a(t))\cot(\frac{k}{H} - \frac{k}{a(t)H})]}{k^2\dot{\phi}^2[a(t)H + k\cot(\frac{k}{H} - \frac{k}{a(t)H})]} \tag{32}$$

4.3 INITIAL CONDITIONS

We were interested in studying the vacuum state, or ground state of the perturbation as well as the excited states. The initial conditions will be given for time $t = 0$, where I will define $m(t = 0) = m_0$ and $w(t = 0) = w_0$. The initial conditions are modeled by the quantum harmonic oscillator. At time $t = 0$ the perturbation wavefunction exists inside the horizon giving $k \gg H_0$ since $a(0) = 1$.

Ground state:

$$\psi_k(\mathcal{R}_k) = \left(\frac{m_0\omega_0}{\pi}\right)^{1/4}\exp\left(-\frac{1}{2}m_0\omega_0\mathcal{R}_k^2\right) \tag{33}$$

Excited states:

$$\psi_k(\mathcal{R}_k)_n = \left(\frac{m_0\omega_0}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(z)\exp\left(-\frac{1}{2}m_0\omega_0\mathcal{R}i_k^2\right) \tag{34}$$

where $z = \sqrt{m_0\omega_0}\mathcal{R}_k^2$ and $H_n(z)$ are the Hermite polynomials.

4.4 TIME-DEPENDENT WAVEFUNCTIONS

Using the method outlined in section 4.1 the time-dependent wavefunctions for the ground state and first excited state are given by:

$$\Psi_k(\mathcal{R}_k, t) = \left(\frac{m_0\omega_0}{\pi}\right)^{1/4}\exp\left(\frac{c_2(t)}{4}\right)\sqrt{\frac{i}{i + c_3(t)m_0\omega_0}}\exp\left[-\frac{1}{2}\frac{i(e^{c_2(t)}m_0\omega_0 - c_1(i + c_3(t)m_0\omega_0))}{i + c_3(t)m_0\omega_0}\mathcal{R}_k^2\right] \tag{35}$$

and for the first excited state:

$$\Psi_k(\mathcal{R}_k, t) = \left(\frac{m_0\omega_0}{\pi}\right)^{1/4} i\exp\left(\frac{3c_2(t)}{4}\right)\sqrt{\frac{i}{i + c_3(t)m_0\omega_0}}\frac{(1 + c_3(t)^2 m_0^2\omega_0^2 - 2e^{c_2(t)}m_0\omega_0\mathcal{R}_k^2}{\sqrt{2}(i + c_3(t)m_0\omega_0)^2}\exp\left[-\frac{1}{2}\frac{i(e^{c_2(t)}m_0\omega_0 - c_1(i + c_3(t)m_0\omega_0))}{i + c_3(t)m_0\omega_0}\mathcal{R}_k^2\right] \tag{36}$$

Now we have a wavefunctional form for the perturbations. The most interesting result here is the fact that the width of the gaussian is the same for both the ground state and the first excited state. Looking at figures (12) and (13) we can see why the Schrodinger picture is more aesthetic in this situation.

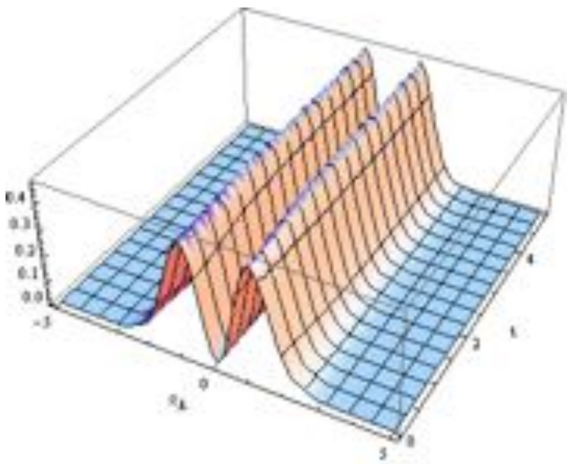

Figure 12: Example of the $|\Psi|^2$ for the first excited state used in these calculations. The scale here is irrelevant, many of the parameters have been set to 1 in order to produce the graph. This is just to give a visual idea of what the Schrodinger picture does.

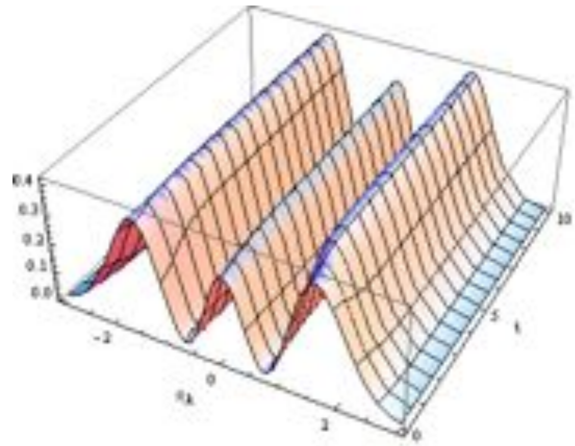

Figure 13: Example of the $|\Psi|^2$ for the second excited state used in these calculations. The scale here is irrelevant, many of the parameters have been set to 1 in order to produce the graph. This is just to give a visual idea of what the Schrodinger picture does.

4.5 CALCULATION OF TWO POINT FUNCTION AND TILT

In calculating the two point functions of the various states we are able to obtain the power spectrum needed [5].

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = (2\pi)^3 \delta(k + k') P_R(k) \tag{37}$$

From this one can easily calculate the scalar spectral index, or tilt (n_s).

$$\Delta_R^2 = \frac{k^3}{2\pi^2} P_R(k) \tag{38}$$

$$n_s - 1 = \frac{d \ln \Delta_R^2}{d \ln k} \tag{39}$$

One can calculate the general form of the power spectrum from the wavefunction in equation (29). For any nth wavefunction and obtain

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle_n (t) = \left(n + \frac{1}{2} \right) \frac{1}{(F_k(t) + F_k^*(t))} \tag{40}$$

Now we have a way to calculate the power spectrum using only the width $F_k(t)$.

From the above wavefunction in equations (35 and 36) an expression for the width of the gaussian ($F_k(t)$) can be obtained.

$$F_k(t) = \frac{i(e^{c_2(t)} m_0 \omega_0 - c_1(t)(i + c_3(t) m_0 \omega_0))}{i + c_3(t) m_0 \omega_0} \tag{41}$$

Now this can be plugged into eq (40) to obtain the general two point function.

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle_n (t) = \left(n + \frac{1}{2} \right) \left(\frac{1 + c_3(t)^2 m_0^2 \omega_0^2}{2 e^{c_2(t)} m_0 \omega_0} \right) \tag{42}$$

As a check, we can also compute the power spectrum and tilt in the sub-horizon limit described by the conditions in equation (14).

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = \left(n + \frac{1}{2} \right) \frac{H^2}{4k\dot{\phi}^2} \tag{43}$$

$$n_s = 3 \tag{44}$$

These are the expected results.

5 RESULTS AND COMPARISON TO PLANCK

Now, we know that at the end of inflation, or for late times, the fluctuations are on super-horizon scales. Therefore we can apply the condition from equation (15). When we do this we obtain a power spectrum at the end of inflation when the fluctuations are outside of the horizon and $\mathcal{R}_k$ is essentially frozen.

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = \left(n + \frac{1}{2} \right) \frac{H^4}{4k^3\dot{\phi}^2} \tag{45}$$

The tilt, or spectral index is the main goal of this work, since it can be compared directly to the value obtained by the Planck results. If we plug in our results from equation (45) into equations (37-39), we can obtain a value for the spectral index.

$$n_s = 1 \tag{46}$$

This is within $\sim 3\sigma$ of the Planck result of $n_s = 0.9603 \pm 0.0073$.

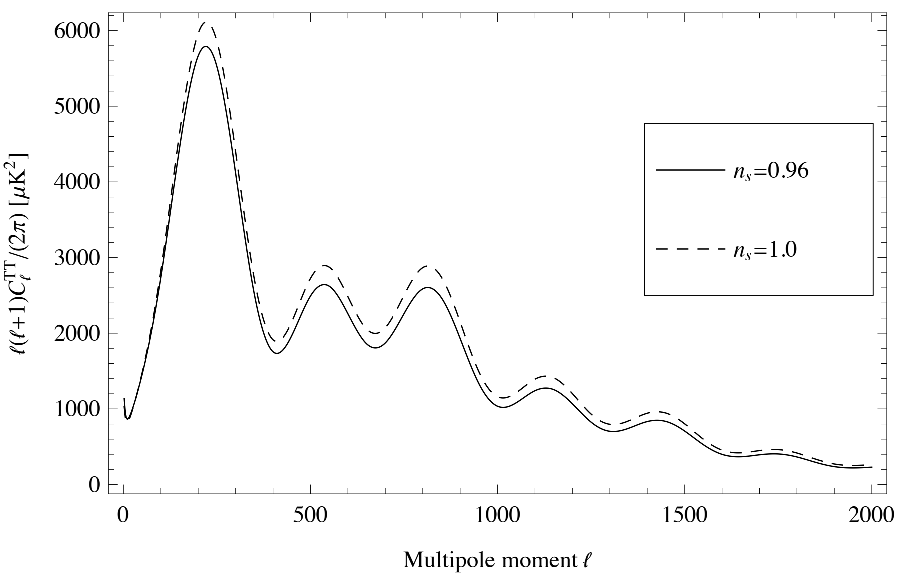

Figure 14: *Angular Power Spectrum* - This shows a comparison between the $n_s = 1$ result and the Planck $n_s = 0.9603$ result. These plots were created using the CAMB web interface on the NASA website [13].

6 CONCLUSION

I have shown that it is possible, under the right assumptions, for certain initial conditions set by a pre-inflationary epoch to have an observable effect (this has already been shown in other papers like [7]).

Furthermore, I have shown that if the initial conditions set in this pre-inflationary epoch are excited quantum states, they too will have an observable effect on the power spectrum. These initial conditions will effectively scale the power spectrum, changing the amplitude, however, the tilt will remain unchanged. This is true for any excited state in a de Sitter Universe.

In calculating the power spectrum and tilt using a de Sitter Universe I found that the tilt will be $n_s = 1$, which is reasonably close to the actual value observed by Planck. However, it is not within the constraints set forth by Planck. The reason for the discrepancy is the fact that we used a de Sitter model which sets $\dot{H}(t) = 0$. This assumption will essentially flatten out the tilt. If we had used a slow-roll model with $\dot{H}(t) \neq 0$ we would have obtained a more accurate value of n_s from the calculations, most likely closer to the actual Planck value.

In order to be able to distinguish a difference between the excited states one would have to look at the higher order fluctuations. As stated in section 4, the Hamiltonian is a leading order approximation of the action, giving a gaussian. To look at higher order terms one could calculate the three point function, which would give information about the non-gaussianity.

References

[1] Freedman, R. A., & Kaufmann III, W. J. (2008). Universe. New York: W.H. Freeman and Company.

[2] Weinberg, S. (2008). Cosmology. Oxford: Oxford University Press.

[3] Kolb, E. W., & Turner, M. S. (1994). The Early Universe. Westview Press.

[4] Komatsu, E. (2011). Some Basics of the Expansion of the Universe, Cosmic Microwave Background, and Large-scale Structure of the Universe. AST396 Lecture Notes. University of Texas at Austin.

[5] Baumann, D. (2009). TASI Lectures on Inflation. Boulder: University of Colorado at Boulder.

[6] Lo, C. (1993). Propagator of the general driven time-dependent oscillator. Physical Review A , 47 (1).

[7] Carney, D., Fischler, W., Paban, S., & Sivanandam, N. (2012). The Inflationary Wavefunction and its Initial Conditions. Journal of Cosmology and Astroparticle Physics.

[8] Meyers, J. (2012). Inflation: Connecting Theory to Observation...

[9] Kirshner, R. (2004). Hubble's diagram and cosmic expansion. Proceedings of the National Academy of Sciences of the United States of America.

[10] Schneider, P. (2006). Extragalactic Astronomy and Cosmology. Berlin Heidelberg New York: Springer.

[11] Planck Collaboration, et.al.,Planck 2013 results. I. Overview of products and scientific results.

[12] Planck Collaboration, et.al.,Planck 2013 results. XVI. Cosmological parameters.

[13] `http://lambda.gsfc.nasa.gov/toolbox/tb_camb_form.cfm`

[14] Maldacena, J. (2003). Non-gaussian features of primordial fluctuations in single field inflationary models. JHEP, 0305(13)

[15] http://abyss.uoregon.edu/~js/lectures/cosmo_101.html

[16] Watson, G. (2011). An Exposition on Inflationary Cosmology. ASTRO-PH (0005003).

A ACKNOWLEDGMENTS

I would like to thank Dr. Sonia Paban for being such a patient advisor during this project. Her vast knowledge of the subject matter and calculations were of great help throughout, and her teaching has given me a new perspective on how to think about theoretical physics. I would also like to thank Dr. Milos Milosavljevic for helping with the Astronomy side of this project. He always asked the right questions to help me understand how and why these calculations could be applied to Astronomy. I would also like to extend my gratitude to Dr. Eiichiro Komatsu. His advice and knowledge of cosmology showed me how to compare theory to observation, and his personal lecture notes on the CMB and cosmology helped me learn a great deal of the conceptual content matter of this project.

Along the way I have also had to obtain help from various people in the Theory Group. Dustin Lorshbough was always willing to help out and make suggestions were needed, and Dr. Ely Kovetz, and his extensive knowledge of inflation was a great help when it came time to compute and compare to Planck. I also greatly appreciate the extensive help and guidance that Dr. Joel Meyers has given me. The use of his Dissertation as a reference was invaluable, and his feed back on my rough draft was extremely helpful. Also, if it wasn't for Joel's advice a few years ago, I might not have even found this project to take on.

Most of all, I would like to thank Dan Carney. He stuck with me throughout the entire year it took to complete this project, meeting with me every week throughout and patiently teaching me everything I needed to know about cosmology, inflation and quantum field theory. It is because of him that this project was a success. My knowledge and experience has grown exponentially due to his guidance.

B SKY MAP

Here I am including a sky map produced by the CAMB program. The first map was created using the my calculated spectral index of $n_s = 1$. For comparison I am including the Planck CMB sky map from earlier in this paper.

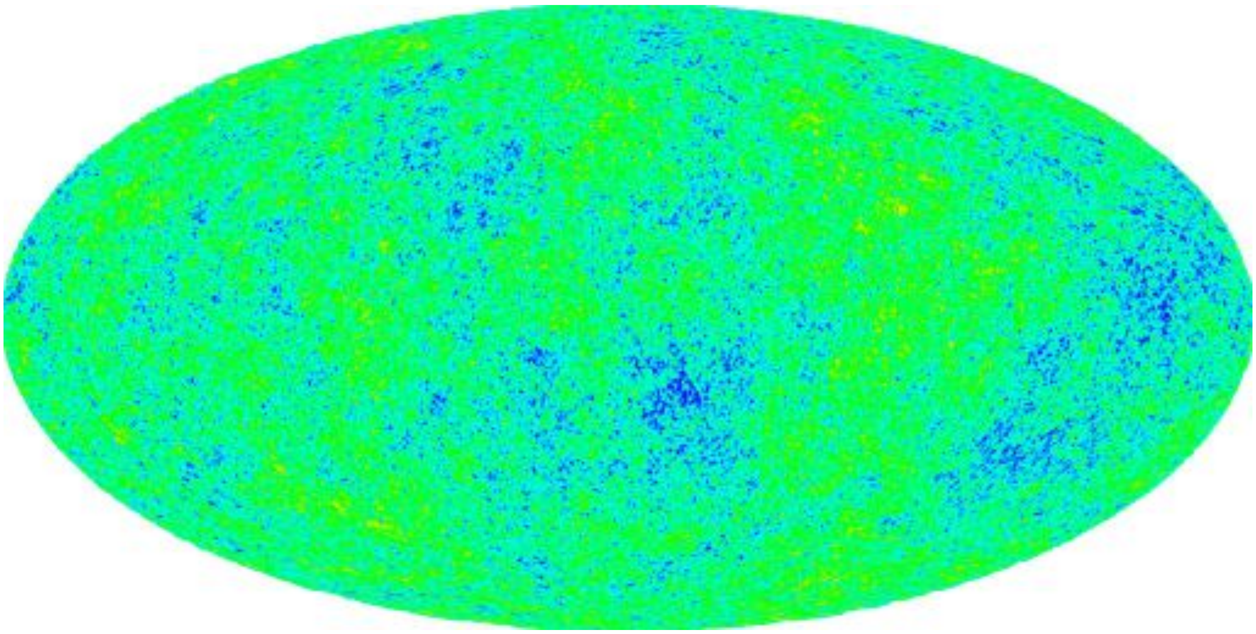

Figure 15: CMB sky map simulated with the CAMB program using the normal parameters and $n_s = 1$.

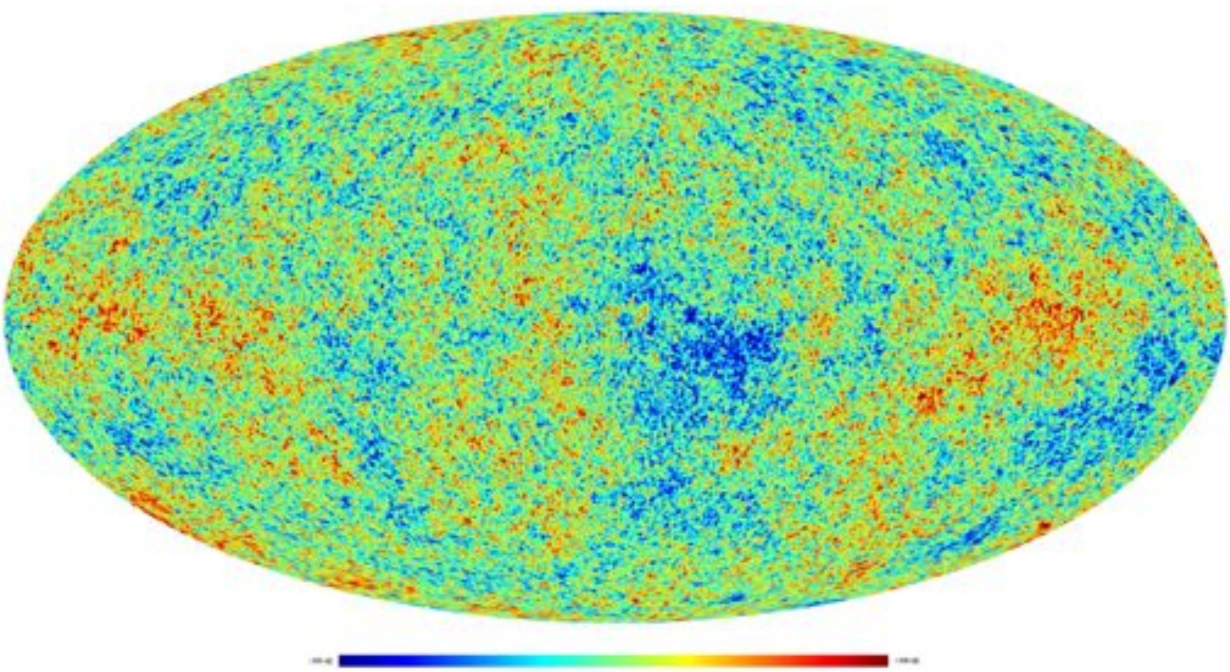

Figure 16: CMB sky map produced by the Planck 2013 results [11].